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On Integral Relations Involving Products of Spheroidal Functions

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ERRATA

Page 6 Eq. (2.9) change $\ldots + d(\xi)\, S^{\mu'}_{\nu'}{}^{(j)}(\xi)\ldots$ to read $\ldots + d(\xi) S^{\mu'}_{\nu'}{}^{(j)'}$

Page 6 Add $(j,k = 1,2,3,4)$ after last eqs.

Page 8 Eq. (2.24) change " α^{μ}_{ν} " to read " $\alpha^{\mu'}_{\nu'}$ ".

Page 9 Eq. (2.26) insert " - " before the braces.

Page 10 Eq. (2.27) change " α^{μ}_{ν} " to read " $\alpha^{\mu'}_{\nu'}$ ".

Eq. (2.28) change " β^{μ}_{ν} " to read " $\beta^{\mu'}_{\nu'}$ ".

Eq. (2.29) change " δ^{μ}_{ν} " to read " $\delta^{\mu'}_{\nu'}$ ".

Page 11 Eq. (2.36) add " $*$ " as superscript to square brackets.

Last line should read " where $\alpha = \alpha^{\mu'}_{\nu'}$, $\beta = \beta^{\mu'}_{\nu'}$, $\delta = \delta^{\mu'}_{\nu'}$.

Page 13 line 14 change " $|\arg(-i\gamma\xi) > \frac{\pi}{2}$ " to read " $|\arg(-i\gamma\xi) < \frac{\pi}{2}$ ".

Page 16 In footnote change " Appendix A" to read " Appendix I".

Page 17 Eq. (4.1) in second integrand change " $Q_n^{-m}(\eta)$ " to read " $Q_n^{-m}(\xi)$ ".

Page 20 line 6 change " $\delta_2^{m-j} = 0$" to read " $\delta_2^{m\pm} = 0$" .

Page 22 Eq. (4.16) change $\dfrac{m\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}}$ to read $\dfrac{m}{\sqrt{(\xi^2-1)(1-\eta^2)}}$

line 2 from bottom, change " $\bar{\beta}_2^{*m\pm}$ " to read " $\bar{\beta}_2^{*m\mp}$ ".

Page 24 line 3 change $-\Psi^{(4)'}_{n+1}$ to read $-\Psi^{(4)}_{n+1}$.

Page 29 Eq. (I.17) chang, " $\alpha^{-m}_{n,p}$ " to read " $\alpha^{m}_{n,p}$ ".

Page 30 Second line after Eq. (I.18) change " Eq. (3.7) or (3.11)" to read Eq. (3.1) and (3.3).

Page 32 Eq. (I.23) change " $\Psi_n^{(j)}(z)$" to read " $\Psi_n^{(j)'}(z)$".

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ON INTEGRAL RELATIONS INVOLVING PRODUCTS OF SPHEROIDAL FUNCTIONS

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Abstract

Starting from a general integral theorem for spheroidal functions, we obtain several integral relations involving products of spheroidal functions by choosing the kernel in the integral to be of the form $e^{-i\mu'\phi}$ Lu, where u is a solution of the time-harmonic wave equation in spheroidal coordinates and L is an operator commuting with the Laplacian. Operators possessing this property are certain linear combinations of the linear and the angular momentum operators and powers and products of these combinations. Several new integral relations between associated Legendre functions have been obtained as limiting cases of the integral relations involving products of spheroidal functions.

Table of Contents

1. Introduction

In this paper we obtain a number of integral relations involving products of spheroidal functions based on a general integral relation due to Ince [1]. We choose the kernel in the integral to be of the form $e^{-iu'\phi}$ Lu, where u is a solution of the time-harmonic wave equation in spheroidal coordinates and L is an operator commuting with the Laplacian. Operators possessing this property are certain linear combinations of the linear and the angular momentum operators and powers and products of the combinations.

Spheroidal functions have played an increasingly important role in problems of applied mathematics and quantum mechanics [2]. For example, the use of these functions has made it possible to find exact solutions of the problems of diffraction of acoustic and of electromagnetic waves by circular discs and apertures [3]. Although a systematic treatment of the theory and applications of spheroidal functions has recently been published by Meixner and Schäfke [4], the theory of spheroidal functions in general has not been as extensively developed as the theory of other special functions of mathematical physics (Legendre, Bessel, Laguerre, Hermite functions, etc.). For the latter functions, whose differential equations are of the hypergeometric type, there exist well-known recurrence relations connecting three contiguous functions. These formulas provide a method of constructing numerical tables, and furnish valuable insight into the nature of the functions themselves.

Whittaker [5] has given a method by means of which he derived 'recurrence relations' for Mathieu functions. This method was used by Sharma [6] for Lamé functions and recently by Marx [7] for spheroidal functions. It should be noted that Whittaker's method does not yield explicit expressions for the coefficients in these 'recurrence relations'. Subsequently Marx [8]

was able to obtain explicit expressions for the coefficients of the 'recurrence relations' for the spheroidal functions by making use of the expansion theorems and the orthogonality properties of these functions.

However, Meixner[9] has shown that the coefficients in the 'recurrence relations' for Mathieu functions given by Whittaker cannot be reduced to simple expressions in the variables and in the parameters as is the case for functions of the hypergeometric type; in fact these coefficients involve products of the functions themselves. Therefore, although Whittaker's formulas for the Mathieu functions have the form of recurrence relations they do not have the properties of recurrence relations for functions of the hypergeometric type.

The procedure used in the present paper is similar to that used by Meixner and Schäfke[4] for Mathieu functions. The different integral relations are obtained by choosing different kernels and taking various solutions of the spheroidal equations. Among the relations thus obtained some yield just the coefficients of the 'recurrence relations' for spheroidal functions. As in the case of Mathieu functions, these coefficients cannot be reduced to expressions which do not involve products of the functions themselves. Marx[8] obtained some of the integral relations derived here. However, our procedure enables us to avoid the use of expansion theorems for spheroidal functions, and our results are more general.

We have also investigated the limiting case, when $\gamma = 0$, which leads to a number of new integral relations involving products of associated Legendre functions.

2. The wave equation in spheroidal coordinates

The three-dimensional time-harmonic wave equation

$$(2.1) \qquad \Delta u + k^2 u = 0,$$

where k is a constant, can be separated in prolate spheroidal coordinates. These are defined by

$$(2.2) \qquad \begin{aligned} x &= c \sqrt{(\xi^2-1)(1-\eta^2)} \cos \phi, \\ y &= c \sqrt{(\xi^2-1)(1-\eta^2)} \sin \phi, \\ z &= c\, \xi\, \eta . \end{aligned}$$

If we take a solution of (2.1) of the form

$$(2.3) \qquad u(x,y,z) = U(\xi)\, V(\eta)\, W(\phi),$$

then the wave equation reduces to three ordinary differential equations

$$(2.4a) \qquad \left[(1-\xi^2)\, U_\xi\right]_\xi + \left[\lambda - \frac{\mu^2}{1-\xi^2} + \gamma^2(1-\xi^2)\right] U = 0,$$

$$(2.4b) \qquad \left[(1-\eta^2)\, V_\eta\right]_\eta + \left[\lambda - \frac{\mu^2}{1-\eta^2} + \gamma^2(1-\eta^2)\right] V = 0,$$

$$(2.4c) \qquad W_{\phi\phi} + \mu^2 W = 0,$$

where

$$(2.5) \qquad \gamma^2 = c^2 k^2$$

and μ and λ are the separation parameters.

The differential equations (2.4a) and (2.4b) are identical; they are called the differential equations of spheroidal functions. They have been dealt with in great detail in the treatise by Meixner and Schäfke mentioned above.

We have the following theorem:

Theorem I. If Γ is a path in the complex η-plane and D is a domain in the ξ-plane and if $u_2(\eta)$ and $u_3(\phi)$ are solutions of (2.4b) and (2.4c) respectively, and furthermore if

(A) $$u(\xi,\eta,\phi) = G(\xi,\eta)\, u_3(\phi)$$

is a regular analytic solution of (2.1) in ξ and η in their respective domains, then

(B) $$u_1(\xi) = \int_\Gamma G(\xi,\eta)\, u_2(\eta)\, d\eta$$

is a solution of (2.4a) in D, provided that

(C) $$(1-\eta^2)\left[\frac{\partial G(\xi,\eta)}{\partial \eta}\, u_2(\eta) - G(\xi,\eta)\,\frac{\partial u_2(\eta)}{\partial \eta}\right]$$

takes the same values at both ends of the path of integration Γ and is such that the indefinite integral of (B) is uniformly convergent for all ξ in D.*

Many known integral relations between spheroidal functions follow in a straightforward way from this theorem if one simply chooses for $u(\xi,\eta,\phi)$ elementary solutions of the wave equation with angular dependence $e^{i\mu\phi}$. Of course one could use the separated solution (2.3) and take as a kernel $G(\xi,\eta)$ in (B)

(2.6) $$G(\xi,\eta) = u(\xi,\eta,\phi)\, e^{-i\mu\phi} = U(\xi)\, V(\eta),$$

---

* A simple proof is given in section 1.133 of ref. [4].

where $U(\xi)$ and $V(\eta)$ are solutions of (2.4a) and (2.4b); but then the integral relation turns out to be trivial. This is, however, not the case if we take Lu instead of u, where L is an operator which commutes with Δ and is such that it produces an angular dependence $e^{i\mu'\phi}$. Then Lu is also a solution of the wave equation, and by taking $e^{-i\mu'\phi}$ Lu as a kernel in (B) one gets non-trivial integral relations involving products of spheroidal functions.

Certain combinations of the components of the linear and angular momentum operators have the property assigned to the operator L. These are

$$
\begin{aligned}
&L_{\pm} = \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} , \\
&L_z = \frac{\partial}{\partial z} , \\
(2.7)\qquad &M_{\pm} = M_x \pm i M_y = (y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}) \pm i (z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}) , \\
&M_z = (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) .
\end{aligned}
$$

Powers and products of these operators possess, of course, the same property as the operators themselves.

We also have the following lemma.

<u>Lemma I</u>. Let $S_\nu^{\mu(j)}(\xi)$ and $S_{\nu'}^{\mu'(j)}(\xi)$ $(j=1,2,3,4)$ be solutions of the spheroidal equation (2.4a)*; then for $\lambda = \lambda_\nu^\mu(\gamma^2)$ and $\lambda = \lambda_{\nu'}^{\mu'}(\gamma^2)$ respectively there exist relations of the type**

$$(2.8)\qquad a(\xi)\, S_\nu^{\mu(j)}(\xi) + b(\xi)\, S_\nu^{\mu(j)'}(\xi) = S_{\nu'}^{\mu'(j)}(\xi),$$

---

* For the definition of the functions $S_\nu^{\mu(j)}(\xi)$ see ref. [4], sec. 3.65. Any two of these functions are linearly independent.

** For $\nu-\nu'$ = integer, $\mu-\mu'$ = integer, these relations can be considered as 're-currence relations' between spheroidal functions; this case has been treated in detail by Marx.

$$(2.9) \qquad c(\xi)\, S_{\nu'}^{\mu'(j)}(\xi) + d(\xi)\, S_{\nu'}^{\mu'(j)}(\xi) = S_{\nu}^{\mu(j)}(\xi),$$

valid for $j = 1,2,3,4$. The coefficients $a(\xi),\ldots\ldots d(\xi)$ are given by

$$(2.10) \qquad a(\xi) = [\mu', \nu'; \mu, \nu]^*, \qquad b(\xi) = [\mu'\nu'; \mu, \nu]$$

$$(2.11) \qquad c(\xi) = [\mu, \nu; \mu', \nu']^*, \qquad d(\xi) = [\mu, \nu; \mu'\nu'] .$$

The expressions in brackets in (2.10) and (2.11) stand for

$$(2.12a) \qquad [\mu', \nu'; \mu, \nu]^* = \frac{S_{\nu'}^{\mu'(j)}(\xi) S_{\nu}^{\mu(k)'}(\xi) - S_{\nu'}^{\mu'(k)}(\xi) S_{\nu}^{\mu(j)'}(\xi)}{W_{\mu\nu}^{(j,k)}(\xi)},$$

$$(2.12b) \qquad [\mu', \nu'; \mu, \nu] = \frac{S_{\nu'}^{\mu'(j)}(\xi) S_{\nu}^{\mu(k)}(\xi) - S_{\nu'}^{\mu'(k)}(\xi) S_{\nu}^{\mu(j)}(\xi)}{W_{\mu\nu}^{(k,j)}(\xi)}$$

and

$$(2.13a) \qquad [\mu, \nu; \mu', \nu']^* = \frac{S_{\nu}^{\mu(j)}(\xi) S_{\nu'}^{\mu'(k)'}(\xi) - S_{\nu}^{\mu(k)}(\xi) S_{\nu'}^{\mu'(j)'}(\xi)}{W_{\mu'\nu'}^{(j,k)}(\xi)},$$

$$(2.13b) \qquad [\mu, \nu; \mu', \nu'] = \frac{S_{\nu}^{\mu(j)}(\xi) S_{\nu'}^{\mu'(k)}(\xi) - S_{\nu}^{\mu(k)}(\xi) S_{\nu'}^{\mu'(j)}(\xi)}{W_{\mu'\nu'}^{(k,j)}(\xi)}.$$

These expressions are independent of $\mu, \nu; \mu', \nu'$. $W_{\mu,\nu}^{(j,k)}(\xi)$ is an abbreviation for the Wronskian

$$S_{\nu}^{\mu(j)}(\xi) S_{\nu}^{\mu(k)'}(\xi) - S_{\nu}^{\mu(k)}(\xi) S_{\nu}^{\mu(j)'}(\xi) .$$

The proof is obvious. Furthermore it follows easily that no other relations of the form (2.8) and (2.9) exist.

The operators (2.7) applied to the solution (2.3) of the wave equation lead to the following kernels:

$$(2.14)\qquad G_1(\xi,\eta) = e^{-i\mu\phi}\, L_z\left(S_\nu^{\mu(j)}(\xi)\, \textstyle\sum_\nu^\mu(\eta)\, e^{i\mu\phi}\right),$$

$$(2.15)\qquad G_2(\xi,\eta) = e^{-i(\mu\pm 1)\phi}\, L_\pm\left(S_\nu^{\mu(j)}(\xi)\, \textstyle\sum_\nu^\mu(\eta)\, e^{i\mu\phi}\right),$$

$$(2.16)\qquad G_3(\xi,\eta) = e^{-i(\mu\pm 1)\phi}\, M_\pm\left(S_\nu^{\mu(j)}(\xi)\, \textstyle\sum_\nu^\mu(\eta)\, e^{i\mu\phi}\right).$$

We restrict ourselves in the following to spheroidal functions with $j = 3,4$; this implies no loss of generality because the other cases, $(j = 1,2)$, can be obtained by employing the relations

$$(2.17)\qquad S_\nu^{\mu(3,4)}(\xi) = S_\nu^{\mu(1)}(\xi) \pm i\, S_\nu^{\mu(2)}(\xi).$$

The operators (2.7) may be written in spheroidal coordinates as

$$(2.18)\qquad L_z = \frac{1}{c}\left[\frac{\xi^2-1}{\xi^2-\eta^2}\,\eta\,\frac{\partial}{\partial\xi} + \frac{1-\eta^2}{\xi^2-\eta^2}\,\xi\,\frac{\partial}{\partial\eta}\right],$$

$$(2.19)\qquad L_\pm = \frac{1}{c}\left[\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) e^{\pm i\phi} \pm \frac{i e^{\pm i\phi}}{\sqrt{(\xi^2-1)(1-\eta^2)}}\,\frac{\partial}{\partial\phi}\right],$$

$$M_{\pm} = \pm i \left[\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2} \left(\eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \eta}\right) e^{\pm i\phi} - \frac{i e^{\pm i\phi}\, \xi\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}} \frac{\partial}{\partial \phi} \right]. \tag{2.20}$$

For these three operators respectively we take as functions $U_2(\eta)$

$$U_2(\eta) = \sum\nolimits^{\mu}_{\nu+2p+1}(\eta), \tag{2.21}$$

$$U_2(\eta) = \sum\nolimits^{(\mu\pm 1)}_{\nu+2p+1}(\eta), \tag{2.22}$$

$$U_2(\eta) = \sum\nolimits^{(\mu\pm 1)}_{\nu+2p}(\eta). \tag{2.23}$$

The functions $\sum^{\mu}_{\nu}(\eta)$, $\sum^{(\mu\pm 1)}_{\nu+2p+1}(\eta)$, etc., as well as the functions $S^{\mu(j)}_{\nu}$, are solutions of the spheroidal equations with $\lambda = \lambda^{\mu}_{\nu}(\gamma^2)$, $\lambda = \lambda^{(\mu\pm 1)}_{\nu+2p+1}(\gamma^2)$, etc. Here one should remark that p is an arbitrary number; the notation $\nu' = \nu+2p+1$ and $\nu' = \nu+2p$ is convenient because in the special case (A') (see below) one has to choose p as an integer.

The path of integration Γ in (B) is chosen in such a way that the condition (C) of Theorem I is satisfied. This choice may, of course, also impose conditions on the admissible functions $\sum^{\mu'}_{\nu'}(\eta)$. We then arrive at the integral relations of spheroidal functions

$$a^{\mu}_{\nu}\, e^{(j)}_1 \gamma S^{\mu(j)}_{\nu+2p+1}(\xi) = \frac{1}{c} \left[S^{\mu(j)'}_{\nu}(\xi) \int_{\Gamma} \frac{(\xi^2-1)}{\xi^2-\eta^2} \eta \sum\nolimits^{\mu}_{\nu}(\eta) \sum\nolimits^{\mu}_{\nu+2p+1}(\eta)\, d\eta \right. \tag{2.24}$$
$$\left. + S^{\mu(j)}_{\nu}(\xi) \int_{\Gamma} \frac{1-\eta^2}{\xi^2-\eta^2} \xi \sum\nolimits^{\mu'}_{\nu}(\eta) \sum\nolimits^{\mu}_{\nu+2p+1}(\eta)\, d\eta \right];$$

$$(2.25)\quad \beta^{\mu'}_{\nu'}\, e_2^{(j)} \gamma S^{\mu\pm1}_{\nu+2p+1}(\xi) = \frac{1}{c}\left[S^{\mu(j)'}_{\nu}(\xi) \int_{\Gamma} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \xi \sum^{\mu}_{\nu}(\eta) \sum^{(\mu\pm1)}_{\nu+2p+1}(\eta)\, d\eta \right.$$

$$- S^{\mu(j)}_{\nu}(\xi) \int_{\Gamma} \left[\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \eta \sum^{\mu'}_{\nu}(\eta) \pm \frac{\mu}{\sqrt{(\xi^2-1)(1-\eta^2)}} \cdot \sum^{\mu}_{\nu}(\eta) \right]$$

$$\left. \cdot \sum^{(\mu\pm1)}_{\nu+2p+1}(\eta)\, d\eta \right],$$

where $\mu' = \mu\pm1$, $\nu' = \nu+2p+1$; and

$$\delta^{\mu'}_{\nu'} e_3^{(j)}\, S^{(\mu\pm1)(j)}_{\nu+2p}(\xi) =$$

$$(2.26)\quad = \pm i \left[S^{\mu(j)'}_{\nu}(\xi) \int_{\Gamma} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \eta \sum^{\mu}_{\nu}(\eta) \sum^{\mu\pm1}_{\nu+2p+1}(\eta)\, d\eta \pm i\, S^{\mu(j)}_{\nu}(\xi) \right.$$

$$\left. \left\{ \int_{\Gamma} \left(\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \xi \sum^{\mu(j)'}_{\nu}(\eta) \pm \frac{\mu\xi\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}} \sum^{\mu}_{\nu}(\eta) \right) \cdot \sum^{(\mu\pm1)}_{\nu+2p}(\eta)\, d\eta \right\} \right]$$

where $\mu' = \mu\pm1$, $\nu' = \nu+2p$, and where α^{μ}_{ν}, $\beta^{\mu'}_{\nu'}$ and $\delta^{\mu'}_{\nu'}$ are constant quantities determined from the asymptotic behavior of the kernels G_1, G_2 and G_3

[eqs. (2.14), (2.15) and (2.16)]. One obtains thus

$$(2.27) \qquad a^{\mu}_{\nu} = \frac{1}{c} \int_{\Gamma} \eta \, \Sigma^{\mu}_{\nu}(\eta) \, \Sigma^{\mu}_{\nu+2p+1}(\eta) \, d\eta,$$

$$(2.28) \qquad \beta^{\mu}_{\nu'} = \frac{1}{c} \int_{\Gamma} \sqrt{1-\eta^2} \, \Sigma^{\mu}_{\nu}(\eta) \, \Sigma^{(\mu\pm 1)}_{\nu+2p+1}(\eta) \, d\eta,$$

$$(2.29) \qquad \delta^{\mu}_{\nu'} = \mp i \int_{\Gamma} \left(\sqrt{1-\eta^2} \, \Sigma^{\mu'}_{\nu}(\eta) \pm \frac{\mu\eta}{\sqrt{1-\eta^2}} \, \Sigma^{\mu}_{\nu}(\eta) \right) \Sigma^{(\mu\pm 1)}_{\nu+2p}(\eta) \, d\eta.$$

The $e^{(j)}_{k}$, $(j=1,2,3,4)$, $(k=1,2,3)$, are phase factors depending on ν', ν of the spheroidal functions $S^{\mu(j)}_{\nu}(\xi)$. For instance if we set $(j=3,4)$, we find

$$(2.30) \qquad e^{(3,4)}_{k} = e^{\pm i(p+1)\pi} \qquad (k=1,2,3)$$

where the plus sign is taken when $j=3$ and the minus sign when $j=4$. The phase factors $e^{(1,2)}_{k}$ can be determined from the asymptotic behavior of $S^{\mu(1,2)}_{\nu}(\xi)$ or directly from the relations (2.17).

Application of Lemma I leads then immediately to the following integral relations involving products of spheroidal functions:

$$\frac{1}{c}\int_{\Gamma}\frac{(\xi^2-1)}{\xi^2-\eta^2}\,\eta\,\Sigma_{\nu}^{\mu}(\eta)\,\Sigma_{\nu+2p+1}^{\mu}(\eta)\,d\eta = \alpha\gamma e_1^{(j)}\left[\mu,\nu;\ \mu,\nu+2p+1\right], \tag{2.31}$$

$$\frac{1}{c}\int_{\Gamma}\frac{1-\eta^2}{\xi^2-\eta^2}\,\xi\,\Sigma_{\nu}^{\mu'}(\eta)\,\Sigma_{\nu+2p+1}^{\mu}(\eta)\,d\eta = \alpha\gamma e_1^{(j)}\left[\mu,\nu+2p+1;\mu,\nu\right]^{*}, \tag{2.32}$$

$$\frac{1}{c}\int_{\Gamma}\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\,\xi\,\Sigma_{\nu}^{\mu}(\eta)\,\Sigma_{\nu+2p+1}^{(\mu\pm1)}(\eta)\,d\eta = \beta\gamma e_2^{(j)}\left[\mu,\nu;\mu+1,\nu+2p+1\right], \tag{2.33}$$

$$-\frac{1}{c}\int_{\Gamma}\left\{\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\,\eta\,\Sigma_{\nu}^{\mu'}(\eta) \pm \frac{\mu}{\sqrt{(\xi^2-1)(1-\eta^2)}}\,\Sigma_{\nu}^{\mu}(\eta)\right\}\Sigma_{\nu+2p+1}^{(\mu\pm1)}(\eta)\,d\eta \tag{2.34}$$

$$= \beta\gamma e_2^{(j)}\left[\mu\pm1,\ \nu+2p+1;\ \mu,\nu\right]^{*},$$

$$\pm i\int_{\Gamma}\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\,\eta\,\Sigma_{\nu}^{\mu}(\eta)\,\Sigma_{\nu+2p+1}^{(\mu\pm1)}(\eta)\,d\eta = \delta\ e_3^{(j)}\left[\mu,\nu,\mu\pm1,\nu+2p\right], \tag{2.35}$$

$$\mp i\int_{\Gamma}\left\{\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\,\xi\,\Sigma_{\nu}^{\mu'}(\eta) \pm \frac{\mu\xi\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}}\,\Sigma_{\nu}^{\mu}(\eta)\right\}\Sigma_{\nu+2p}^{(\mu\pm1)}(\eta)\,d\eta \tag{2.36}$$

$$= \delta\ e_3^{(j)}\left[\mu\pm1,\ \nu+2p;\ \mu,\nu\right],$$

where $\alpha = \alpha_{\nu}^{\mu}$, $\beta = \beta_{\nu}^{\mu}$, $\delta = \delta_{\nu}^{\mu}$.

3. Special kernels

The results of Section 2 are general in so far as they are valid for any function $\Sigma_\nu^\mu(\eta)$ which is a solution of the spheroidal equation and for a path Γ satisfying condition (C) of Theorem I. Now, we shall consider specific solutions of the spheroidal equation by restricting the path of integration and the parameters μ, ν and γ.

Case A. Let $(\nu-\mu)$ be an integer p. The spheroidal functions $\Sigma_\nu^\mu(\eta)$ then become $\Sigma_\nu^\mu(\eta) = ps_\nu^\mu(\eta;\gamma^2)$, $\Sigma_{\nu+2p+1}^{(\mu\pm1)}(\eta) = ps_{\nu+2p+1}^{(\mu\pm1)}(\eta;\gamma^2)$, etc. An appropriate path of integration Γ is then the real interval $[-1,1]$. We should remark, however, that $ps_\nu^\mu(\eta;\gamma^2)$ remains finite at ±1 if

$$\operatorname{Re}\mu \leq 0 \qquad \text{for} \qquad \nu+\mu = 0, 1, 2,\cdots$$

or

$$\operatorname{Re}\mu \leq 0 \qquad \text{for} \qquad \nu-\mu = -1, -2, -3,\cdots .$$

If $\operatorname{Re}\mu > 0$, then $ps_\nu^\mu(\eta;\gamma^2)$ will in general be infinite for $\eta = +1$, except when μ and ν are both integers, in which case it remains finite at $\eta = \pm1$. In the latter case, the conditions of Theorem I are fulfilled, as one sees from the fact that the functions $ps_\nu^\mu(\eta;\gamma^2)$ are then even for $\nu-\mu$ even or odd for $\nu-\mu$ odd, and behave as $(1-\eta^2)^{m/2}$ in the neighborhood of $\eta = \pm1$. But there is no loss of generality from the restriction $\operatorname{Re}\mu \leq 0$, because for $\operatorname{Re}\mu > 0$ there exists a certain combination of $ps_\nu^\mu(\eta,\gamma^2)$ and $qs_\nu^\mu(\eta;\gamma^2)$ which is finite for $\nu-\mu = 0, 1, 2,\ldots$ or $\nu+\mu = -1, -2, -3\ldots$, and then it can be expressed by $ps_\nu^{-\mu}(\eta;\gamma^2)$ with $\operatorname{Re}\mu > 0$, so that this case is reduced to the former one.

It is further necessary to distinguish between the cases (2.24), (2.25) and (2.26) and one must check in each particular case whether the condition (C) of Theorem I is satisfied.

Case B. Let $\sum_{\nu}^{\mu}(\eta) = ps_{\nu}^{-\mu}(\eta;\gamma^2)$. For real $\gamma > 0$, we take for Γ the interval $[1,\dots\infty]$; otherwise, we have as an appropriate path Γ

$$\left[1,\dots\infty \arg\frac{1}{\gamma}\right] \quad \text{for} \quad \operatorname{Re}\gamma > 0$$

and

$$\left[1,\dots\infty \arg\gamma^{*}\right] \quad \text{for} \quad \operatorname{Re}\gamma < 0 \,,$$

where γ^{*} is the conjugate complex of γ.

Case C. For $\sum_{\nu}^{\mu}(\eta)$ arbitrary, we choose Γ to be a Jordan-Pochhammer path (double loop around any pair of singular points of the spheroidal equation).

Case D. If we take $\sum_{\nu}^{\mu}(\eta) = S_{\nu}^{\mu(3)}(\eta)$ or $S_{\nu}^{\mu(4)}(\eta)$, then the argument of η for $\eta \to \infty$ has to be chosen in such a way that

$$|\arg(-i\gamma\xi)| > \frac{\pi}{2} \quad \text{or} \quad |\arg(i\gamma\xi)| < \frac{\pi}{2}$$

respectively. The path of integration is then a single loop starting from $i\infty\, e^{-i\omega}$, passing around the point +1 in the negative direction, and returning to infinity in the direction $\infty\cdot e^{3(\pi i - i\omega)/2}$. For $S_{\nu}^{\mu(3)}(\eta)$ and for $S_{\nu}^{\mu(4)}(\eta)$ the path Γ starts at $\infty\cdot e^{3(i\pi - i\omega)/2}$ and after passing around +1 in the negative direction returns to infinity along a line $\infty\cdot e^{-i\frac{\pi}{2}-\omega}$.

(See Figs. 1 and 2.)

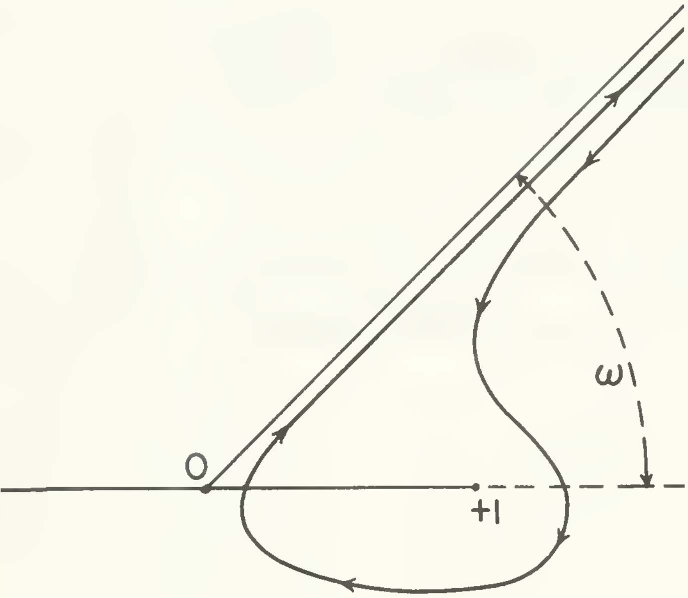

Figure 1

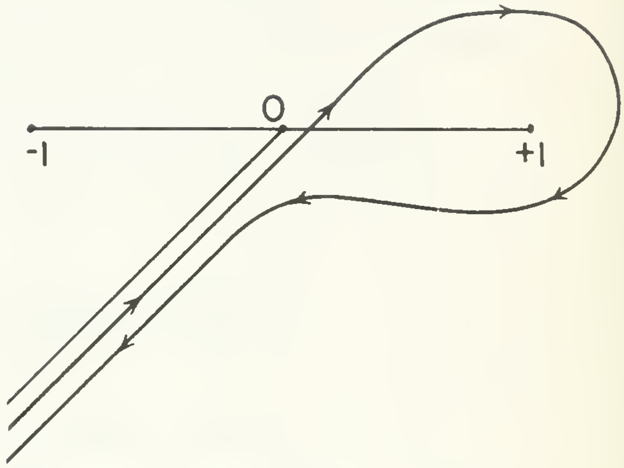

Figure 2

<u>Case E.</u> If we choose $\sum_{\nu}^{\mu}(\eta) = S_{\nu}^{\mu(1)}(\eta)$ then Γ is a loop similar to that in case D, but now the path goes round both points -1 and +1 (see Fig. 3).

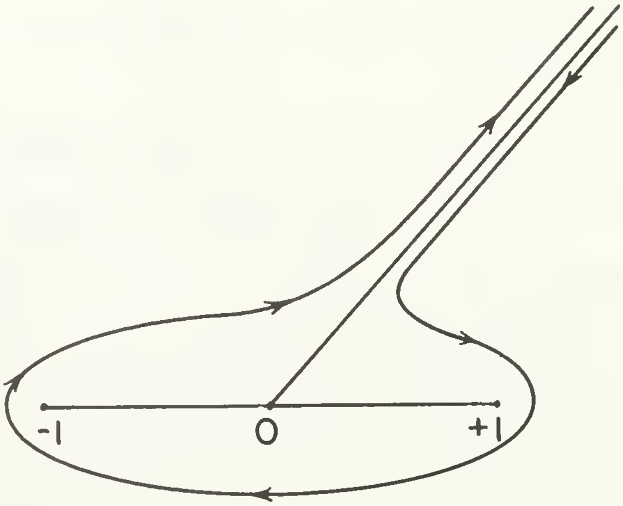

Figure 3

Case A$'$. An important instance of case A occurs when we restrict μ and ν to integral values, $\nu = n$, $\mu = m$ and $n \geq m \geq 0$. In this case the spheroidal functions are either

$$\sum_n^m(\eta) = ps_n^m(\eta;\gamma^2), \qquad \sum_{n+2p+1}^{(m\pm1)}(\eta) = ps_{n+2p+1}^{(m\pm1)}(\eta;\gamma^2),$$

$$\sum_{n+2p}^{(m\pm1)}(\eta) = ps_{n+2p}^{(m\pm1)}(\eta;\gamma^2),$$

or

$$\sum_{n+2p+1}^{(m\pm1)}(\eta) = ps_{n+2p+1}^{-(m\pm1)}(\eta;\gamma^2), \quad \sum_{n+2p}^{(m\pm1)}(\eta) = ps_{n+2p}^{-(m\pm1)}(\eta;\gamma^2).$$

Thus we can expand the spheroidal functions $ps_n^m(\eta;\gamma^2)$, etc., in associated Legendre functions of the form

$$ps_n^m(\eta;\gamma^2) = \sum_{r=-\infty}^{\infty} a_{n,2r}^m(\gamma^2)\, P_{n+2r}^m(\eta), \tag{3.1}$$

$$ps_{n+2p+1}^{-m}(\eta;\gamma^2) = \sum_{r=-\infty}^{\infty} a_{n+2p+1,2r}^{-m}(\gamma^2)\, P_{n+2p+1+2r}^{-m}(\eta). \tag{3.2}$$

The constants α, β, δ are given by the expressions (2.27)-(2.29), which can be easily evaluated from the properties of associated Legendre functions. For instance, we find the following expression for the constant $\alpha = \alpha_{n,p}^m(\gamma^2)$:

$$\alpha = \alpha^m_{n,p}(\gamma^2) = \frac{2(-1)^m}{c(2n+4r+1)} \left[\sum_{r=-\infty}^{\infty} a^m_{n,2r}(\gamma^2)\, a^{-m}_{n+2p+1,2r-2p}(\gamma^2)\, \frac{n+2r-m+1}{2n+4r+3} \right.$$

(3.3)

$$\left. + \sum_{r=-\infty}^{\infty} a^m_{n,2r}(\gamma^2)\, a^{-m}_{n+2p+1,2r-2p-2}(\gamma^2)\, \frac{n+2r+m}{2n+4r+1} \right] .$$

The other constants β, δ can be found in a similar way by substituting for the spheroidal functions $ps^{\mu}_{\nu}(\eta;\gamma^2)$, $ps^{-(\mu+1)}_{\nu+2p+1}$, etc., the corresponding expansions in associated Legendre functions in (2.28) and (2.29) and then performing termwise integration.

4. Limiting cases

There are two limiting cases for which the spheroidal functions reduce to simpler functions. If we put $\gamma = 0$, then the differential equation (2.4a) goes over into the differential equation of the associated Legendre functions. We therefore expect that our integral relations reduce in this limiting case to integral relations between products of associated Legendre functions of the first and second kind.

However, if we introduce $\gamma\xi = z$ as a new variable and then set $\gamma = 0$, the integrands in (2.24), (2.25) and (2.26) split into two factors, one depending on z and the other on η . Therefore this limiting case of our integral relations gives a rather trivial result.

We now shall proceed to work out these limiting cases in detail.

Case I. $\gamma = 0$. It is easier to treat the limiting case in a direct way instead of working out the limiting process from our general results arrived at in the previous sections*. Since

---

* This alternative method is given in Appendix A.

$$ps_n^m(\eta;\gamma^2) \to P_n^m(\eta) \qquad \text{for} \qquad \gamma = 0$$

while $S_n^{m(3,4)}(\xi;\gamma)$ reduces to a linear combination of $P_n^{\pm m}(\xi)$ and $Q_n^{\pm m}(\xi)$, we can at once write down the following formulas*

$$(4.1) \quad a_1 Q_{n+2p+1}^{-m}(\xi) + \bar{a}_1 P_{n+2p+1}^{-m}(\xi) = \int_{-1}^{1} \frac{\xi^2-1}{\xi^2-\eta^2}\, \eta\, Q_n^{-m'}(\xi)\, P_n^m(\eta)\, P_{n+2p+1}^{-m}(\eta)\, d\eta$$

$$+ \int_{-1}^{1} \frac{1-\eta^2}{\xi^2-\eta^2}\, \xi\, Q_n^{-m}(\eta)\, P_n^{m'}(\eta)\, P_{n+2p+1}^{-m}(\eta)\, d\eta,$$

and

$$(4.2) \quad a_2 Q_{n+2p+1}^{-m}(\xi) + \bar{a}_2 P_{n+2p+1}^{-m}(\xi) = \int_{-1}^{1} \frac{\xi^2-1}{\xi^2-\eta^2}\, \eta\, P_n^{-m'}(\xi)\, P_n^m(\eta)\, P_{n+2p+1}^{-m}(\eta)\, d\eta$$

$$+ \int_{-1}^{1} \frac{1-\eta^2}{\xi^2-\eta^2}\, \xi\, P_n^{-m}(\xi)\, P_n^{m'}(\eta)\, P_{n+2p+1}^{-m}(\eta)\, d\eta,$$

where the coefficients a_1, $\bar{a}_1$; a_2, $\bar{a}_2$ are to be determined. From the behavior for large ξ of the integrals and of the functions on the left side of (4.1) and (4.2) one immediately concludes that $\bar{a}_1 = 0$ and $a_2 = 0$ and furthermore that

$$(4.3) \qquad a_1 = 0 \quad \text{for} \quad p < 0, \qquad \bar{a}_2 = 0 \quad \text{for} \quad p \geq 0.$$

---

* Similar expressions can be written for $P_n^m(\xi)$ and $Q_n^m(\xi)$ (see remark following Eq. (4.18)).

In the other cases one finds, expanding in descending powers of ξ and comparing the coefficients of the leading terms,

$$(4.4) \qquad \alpha_1 = 2(-1)^{m+1} \quad (p \geq 0), \qquad \bar{\alpha}_2 = 2(-1)^m \quad (p < 0) .$$

Solving for the two integrals occurring in (4.1) and (4.2), we get

$$(4.5) \qquad \alpha_1 Q^{-m}_{n+2p+1}(\xi)\, P^m_n(\xi) - \bar{\alpha}_2 P^{-m}_{n+2p+1}(\xi)\, Q^m_n(\xi) =$$

$$= (-1)^{m+1} \int_{-1}^{1} \frac{\eta}{\xi^2-\eta^2} P^m_n(\eta)\, P^{-m}_{n+2p+1}(\eta)\, d\eta ,$$

$$(4.6) \qquad \alpha_1 Q^{-m}_{n+2p+1}(\xi)\, P^{m'}_n(\xi) - \bar{\alpha}_2 P^{-m}_{n+2p+1}(\xi)\, Q^{m'}_n(\xi) =$$

$$= (-1)^m \int_{-1}^{1} \frac{1-\eta^2}{\xi^2-\eta^2} P^{m'}_n(\eta)\, P^{-m}_{n+2p+1}(\eta)\, d\eta .$$

In a similar way we get from (2.25) in the limiting case

$$(4.7) \qquad \beta_1^{m\pm} Q^{-(m\pm1)}_{n+2p+1}(\xi) + \bar{\beta}_1^{m\pm} P^{-(m\pm1)}_{n+2p+1}(\xi) =$$

$$= \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \xi\, Q^{-m'}_n(\xi)\, P^m_n(\eta)\, P^{-(m\pm1)}_{n+2p+1}(\eta)\, d\eta$$

$$- \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \eta\, Q^{-m}_n(\xi)\, P^{m'}_n(\eta)\, P^{-(m\pm1)}_{n+2p+1}(\eta)\, d\eta$$

$$\mp \int \frac{m}{\sqrt{(\xi^2-1)(1-\eta^2)}}\, Q^{-m}_n(\xi)\, P^m_n(\eta)\, P^{-(m\pm1)}_{n+2p+1}(\eta)\, d\eta ,$$

$$(4.8)\quad \beta_2^{m\pm}\, Q_{n+2p+1}^{-(m\pm1)}(\xi) + \bar{\beta}_2^{m\pm}\, P_{n+2p+1}^{-(m\pm1)}(\xi) =$$

$$= \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \xi\, P_n^{-m'}(\xi)\, P_n^m(\eta)\, P_{n+2p+1}^{-(m\pm1)}(\eta)\, d\eta$$

$$- \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \eta\, P_n^{-m}(\xi)\, P_n^{m'}(\eta)\, P_{n+2p+1}^{-(m\pm1)}(\eta)\, d\eta$$

$$\mp \int_{-1}^{1} \frac{m}{\sqrt{(\xi^2-1)(1-\eta^2)}}\, P_n^{-m}(\xi)\, P_n^m(\eta)\, P_{n+2p+1}^{-(m\pm1)}(\eta)\, d\eta ,$$

while (2.26) reduces to

$$(4.9)\quad \delta_1^{m\pm}\, Q_{n+2p+1}^{-(m\pm1)}(\xi) + \bar{\delta}_1^{m\pm}\, P_{n+2p+1}^{-(m\pm1)}(\xi) =$$

$$= \pm i \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \eta\, Q_n^{-m'}(\xi)\, P_n^m(\eta)\, P_{n+2p}^{-(m\pm1)}(\eta)\, d\eta$$

$$\mp i \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \xi\, Q_n^{-m}(\xi)\, P_n^{m'}(\eta)\, P_{n+2p}^{-(m\pm1)}(\eta)\, d\eta$$

$$-im \int_{-1}^{1} \frac{\xi\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}}\, Q_n^{-m}(\xi)\, P_n^m(\eta)\, P_{n+2p}^{-(m\pm1)}(\eta)\, d\eta$$

and

$$(4.10)\quad \delta_2^{m\pm}\, Q_{n+2p}^{-(m\pm 1)}(\xi) + \bar{\delta}_2^{m\pm}\, P_{n+2p}^{-(m\pm 1)}(\xi) =$$

$$= \pm i \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \eta\, P_n^{-m'}(\xi)\, P_n^m(\eta)\, P_{n+2p}^{-(m\pm 1)}(\eta)\, d\eta$$

$$\mp i \int_{-1}^{1} \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\, \xi\, P_n^{-m}(\xi)\, P_n^{m'}(\eta)\, P_{n+2p}^{-(m\pm 1)}(\eta)\, d\eta$$

$$-im \int_{-1}^{1} \frac{\xi\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}}\, P_n^{-m}(\xi)\, P_n^m(\eta)\, P_{n+2p}^{-(m\pm 1)}(\eta)\, d\eta\, .$$

Following the same arguments as for the α_j and β_j $(j=1,2)$, we find $\bar{\beta}_1^{m\pm} = 0$ and $\beta_2^{m\pm} = 0$ and $\bar{\delta}_1^{m\pm} = 0$ and $\delta_2^{m-j} = 0$. Furthermore, one has

$$(4.11)\qquad \beta_1^{m\pm} = 0 \quad \text{for} \quad p < 0, \qquad \bar{\beta}_2^{m\pm} = 0 \quad \text{for} \quad p \geq 0$$

and

$$(4.12)\qquad \delta_1^{m\pm} = 0 \quad \text{for} \quad p \leq 0, \qquad \bar{\delta}_2^{m\pm} = 0 \quad \text{for} \quad p > 0\, .$$

In all other cases one finds, expanding in descending powers of ξ and comparing the coefficients of the leading terms,

$$(4.13a)\qquad \beta_1^{m+} = -\frac{n+m+2}{2n+1}(-1)^{m+1}, \qquad \bar{\beta}_2^{m+} = -\frac{n-m-1}{2n+1}$$

$$(4.13b)\qquad \beta_1^{m-} = \frac{n-m+2}{2n+1}(-1)^{m-1}, \qquad \bar{\beta}_2^{m-} = \frac{n+m-1}{2n+1}$$

$$(4.14a)\qquad \delta_1^{m+} = i\,\frac{n+m+2}{2n+1}(-1)^{m+1}, \qquad \bar{\delta}_2^{m+} = -i\,\frac{n-m-1}{2n+1}$$

$$(4.14b)\qquad \delta_1^{m-} = -i\,\frac{n-m+2}{2n+1}(-1)^{m-1}, \qquad \bar{\delta}_2^{m-} = -i\,\frac{n+m-1}{2n+1}\,.$$

Solving for the two integrals occurring in (4.7) and (4.8), we get

$$(4.15)\qquad \beta_1^{m\pm}\, Q_{n+2p+1}^{-(m\pm1)}(\xi)\, P_n^m(\xi) - \bar{\beta}_2^{m\pm}\, P_{n+2p+1}^{-(m\pm1)}(\xi)\, Q_n^m(\xi) =$$

$$= -\frac{1}{\sqrt{\xi^2-1}} \int_{-1}^{1} \frac{\sqrt{1-\eta^2}}{\xi^2-\eta^2}\, \xi\, P_n^m(\eta)\, P_{n+2p+1}^{-(m\pm1)}(\eta)\, d\eta,$$

$$(4.16)\quad \beta_1^{m\pm} Q_{n+2p+1}^{-(m\pm 1)}(\xi)\, P_n^{m'}(\xi) - \overline{\beta}_2^{m\pm} P_{n+2p+1}^{-(m\pm 1)}(\xi)\, Q_n^m(\xi) =$$

$$= \frac{\xi}{\xi^2-1}\int_{-1}^{1}\left[\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\,\eta\, P_n^{m'}(\eta) \mp \frac{m\eta}{\sqrt{(\xi^2-1)(1-\eta^2)}}\,P_n^m(\eta)\right] P_{n+2p+1}^{-(m\pm 1)}(\eta)\,d\eta,$$

and from (4.9) and (4.10) one obtains

$$(4.17)\quad \delta_1^{m\pm} Q_{n+2p}^{-(m\pm 1)}(\xi)\, P_n^m(\xi) - \overline{\delta}_2^{m\pm} P_{n+2p}^{-(m\pm 1)}(\xi)\, Q_n^m(\xi) =$$

$$= \mp \frac{i}{\sqrt{\xi^2-1}}\int_{-1}^{1}\frac{\sqrt{1-\eta^2}}{\xi^2-\eta}\,\eta\, P_n^m(\eta)\, P_{n+2p}^{-(m\pm 1)}(\eta)\,d\eta$$

and

$$\delta_1^{m\pm} Q_{n+2p}^{-(m\pm 1)}(\xi)\, P_n^{m'}(\xi) - \overline{\delta}_2^{m} P_{n+2p}^{-(m\pm 1)}(\xi)\, Q_n^{m'}(\xi) =$$

$$(4.18)\quad = \pm \frac{i\xi}{\xi^2-1}\int_{-1}^{1}\left(\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2}\,P_n^{m'}(\eta) - \frac{m\eta}{\sqrt{(\xi-1)(1-\eta^2)}}\,P_n^m(\eta)\right) P_{n+2p}^{-(m\pm 1)}(\eta)\,d\eta.$$

Finally we wish to remark that if we replace m by -m and $-(m\pm 1)$ by $(m\mp 1)$ in the associated Legendre functions one gets similar integral expressions which involve associated Legendre functions and whose coefficients $\beta_1^{*m\mp}$, $\overline{\beta}_2^{*m\pm}$; $\delta_1^{*m\mp}$, $\overline{\delta}_2^{*m\mp}$ are obtained from those of (4.13a), (4.14b) by replacing m by -m.

<u>Case II.</u> Let us replace ξ by z/γ and then put $\gamma = 0$. The results of Sections 2 and 3 are easily applied to this case by replacing $ps_n^m(\eta;\gamma^2)$ by $P_n^m(\eta)$ and $S_n^{m(j)}(\xi)$ by $\Psi_n^{(j)}(z)$. The functions $\Psi_n^{(j)}(z)$ are defined by

$$\Psi^{(j)}(z) = \sqrt{\frac{\pi}{2z}} \quad Z^{(j)}_{n+1/2}(z) \tag{4.19}$$

where for $j = 1,2,3,4$ respectively, $Z^{(j)}_{n+1/2}(z)$ is the spherical Bessel function, the Neumann function, and the Hankel function of the first and the second kind. The remaining task is the evaluation of the constants $\alpha^m_{n,p}$, $\beta^{m\pm1}_{n,p}$, and $\delta^{m\pm1}_{n,p}$ for $\gamma = 0$. One finds

$$\alpha^m_{n,o} = \frac{2}{2n+1} \; \frac{n-m+1}{2n+1} \;,$$

$$\alpha^m_{n,-1} = \frac{2}{2n-1} \; \frac{n+m}{2n+1} \;;$$

$$\beta^{m+1}_{n,o} = \frac{2}{2n+1} \; \frac{1}{2n+3} \;, \qquad \beta^{m+1}_{n,-1} = -\frac{2}{2n+1} \; \frac{1}{2n-1} \;,$$

$$\beta^{m-1}_{n,o} = -\frac{2}{2n+1} \; \frac{(n-m+1)(n-m+2)}{2n+3} \;, \qquad \beta^{m-1}_{n-1} = \frac{2}{2n+1} \; \frac{(n+m)(n+m-1)}{2n-1} \;;$$

$$\delta^{m+1}_{n,o} = \frac{n-m+1}{(2n+1)(n+m+1)} \;,$$

$$\delta^{m-1}_{n,o} = \frac{(n+2n-2)(n-m+1)}{2n+1} \;.$$

All the other coefficients vanish. Finally, the expressions (2.12a) and (2.1b) reduce for $j = 3,4$ to

$$\Psi_{n+1}^{(3)}(z)\, \Psi_n^{(4)'}(z) - \Psi_{n+1}^{(4)'}(z)\, \Psi_n^{(3)'}(z) = \frac{n[4,3]}{z},$$

$$\Psi_{n-1}^{(3)}(z)\, \Psi_n^{(4)'}(z) - \Psi_{n-1}^{(4)}(z)\, \Psi_n^{(3)'}(z) = \frac{(n+1)[4,3]}{z}$$

and

$$\Psi_{n+1}^{(3)}(z)\, \Psi_n^{(4)}(z) - \Psi_{n+1}^{(4)}(z)\, \Psi_n^{(3)}(z) = -\frac{[4,3]}{z}$$

$$\Psi_{n-1}^{(3)}(z)\, \Psi_n^{(3)}(z) - \Psi_{n-1}^{(4)}(z)\, \Psi_n^{(3)}(z) = \frac{[4,3]}{z},$$

where $[3,4]$ is the Wronskian $\Psi_n^{(3)}(z)\, \Psi_n^{(4)'}(z) - \Psi_n^{(4)}(z)\, \Psi_n^{(3)'}(z)$. These last results are rather trivial.

Acknowledgement

I wish to thank Professor Josef Meixner of the University of Aachen for his helpful interest.

Appendix I. Integral relations involving products of associated Legendre functions

In Section 4 we derived integral relations between associated Legendre functions. Here we shall give an alternative method of deriving such integral relations by considering the general formulas (2.34), (2.36) for γ small and then taking the limit as $\gamma \to 0$.

In general we can write (2.31) and (2.32) in the form

$$\text{(I.1)} \qquad a^t_{n,p}\, \gamma^2\, e^{\pm i(p+1)\pi} \left[S^{t(3,4)}_{n+2p+1}(\xi)\, S^{m(4,3)}_{n}(\xi) - S^{t(4,3)}_{n+2p+1}(\xi)\, S^{m(3,4)}_{n}(\xi) \right]$$

$$= a^t_{n,p}(\xi)\, [4,3]^{*},$$

$$\text{(I.2)} \qquad a^t_{n,p}\, \gamma^2\, e^{\pm i(p+1)\pi} \left[S^{t(3,4)}_{n+2p+1}(\xi)\, S^{m(4,3)'}_{n}(\xi) - S^{t(4,3)}_{n+2p+1}(\xi)\, S^{m(3,4)'}_{n}(\xi) \right]$$

$$= b^{(z)}_{n,\gamma}(\xi)\, [4,3] ,$$

where $\mu = m$, $\nu = n$ and $\mu' = t$, $\nu' = n+2p+1$ or $n + 2p$. We can replace $S^{t(3,4)}_{\nu}(\xi)$ $S^{t(1)}_{\nu}(\xi)$ using the relations

$$\text{(I.3)} \qquad \cos(\nu\pi) S^{t(3)}_{\nu}(\xi) = -i\, S^{t(1)}_{-\nu-1}(\xi) + e^{-i\nu\pi}\, S^{t(1)}_{\nu}(\xi),$$

---

* We denote the Wronskian of the spheroidal functions by [3,4].

$$\cos(\nu\pi) S_\nu^{t(4)}(\xi) = i\, S_{-\nu-1}^{t(1)}(\xi) + e^{+i\nu\pi} S_\nu^{t(1)}(\xi) \quad . \tag{I.4}$$

We introduce the abbreviation

$$A = \left[S_{n+2p+1}^{t(3)}(\xi)\, S_n^{m(4)}(\xi) - S_{n+2p+1}^{t(4)}(\xi)\, S_n^{m(3)}(\xi) \right] , \tag{I.5}$$

or, in terms of $S_\nu^{t(1)}(\xi)$,

$$A = 2i(-1)^n \left[S_{-n-2p-2}^{t(1)}(\xi)\, S_n^{m(1)}(\xi) - S_{n+2p+1}^{t(1)}(\xi)\, S_{-\nu-1}^{m(1)}(\xi) \right] . \tag{I.6}$$

Writing

$$S_\nu^{t(1)}(\xi;\gamma) = K_\nu^t(\gamma)\, Ps_\nu^t(\xi;\gamma^2) , \tag{I.7}$$

$$S_{-\nu-1}^{t(1)}(\xi;\gamma) = \frac{e^{-it\pi}\cos(\nu\pi)\,\Gamma(\nu-t+1)}{\gamma K_\nu^t(\gamma) A_\nu^t(\gamma^2) A_\nu^{-t}(\gamma^2)\,\Gamma(\nu+t+1)} \cdot Qs_\nu^t(\xi;\gamma^2) \tag{I.8}$$

and substituting in (I.6), we find

$$A = 2i(-1)^m \left[\frac{\Gamma(n-t+1) V_n^m(\gamma) V_{-n-2p-2}^t(\gamma^2)}{\Gamma(n+2p+1-t+1)} \cdot Qs_{n+2p+1}^{-t}(\xi;\gamma^2)\, Ps_n^m(\xi;\gamma^2) \right. \tag{I.9}$$

$$\left. + \frac{\Gamma(n+2p+1+t+1)}{\Gamma(n+t+1)} V_{n+2p+1}^t(\gamma)\, V_{-n-1}^m(\gamma)\, Ps_{n+2p+1}^{-t}(\xi;\gamma^2)\, Qs_n^m(\xi;\gamma^2) \right] ,$$

where

$$\text{(I.10)} \qquad K_\nu^t(\gamma) = \cos(\nu\pi)\,\Gamma(\nu-t+1)\,V_\nu^t(\gamma)$$

and

$$\text{(I.11)} \qquad \gamma\, V_\nu^t(\gamma)\, V_{-\nu-1}^{t}(\gamma)\, A_\nu^t(\gamma^2)\, A_\nu^{-t}(\gamma^2) = 1.$$

For small values of γ the function $V_\nu^t(\gamma)$ is given by the expression*

$$\text{(I.12)} \qquad V_\nu^t(\gamma) = \frac{1}{2}\left(\frac{\gamma}{4}\right)^\nu \frac{\Gamma(\frac{1}{2}-\nu)}{\Gamma(\nu+\frac{3}{2})}\left(1 + 0\,(\gamma^2)\right).$$

Substituting this expression into (I.9) we obtain for small values of γ

$$\text{(I.13)} \quad A = 2i(-1)^m \left[\frac{\Gamma(n-t+1)\,\Gamma(\frac{1}{2}-n)\,\Gamma(n+2p+\frac{5}{2})}{4\left(\Gamma(n+2p+1-t+1)\,\Gamma(n+\frac{3}{2})\,\Gamma(-n-2p-\frac{1}{2})\right)} \left(\frac{\gamma}{4}\right)^{-2p-2} \cdot Qs_{n+2p+1}^{-t}(\xi;\gamma^2)\, Ps_n^m(\xi;\gamma^2) \right.$$

$$\left. + \frac{\Gamma(n+2p+1+t+1)\,\Gamma(n+\frac{3}{2})\,\Gamma(-n-2p-\frac{1}{2})}{4\left(\Gamma(n+t+1)\,\Gamma(\frac{1}{2}-n)\,\Gamma(n+2p+\frac{5}{2})\right)} \left(\frac{\gamma}{4}\right)^{2p} \cdot Ps_{n+2p+1}^{-t}(\xi;\gamma^2)\, Qs_n^m(\xi;\gamma^2) \right]$$

---

* See ref. [4], page 297.

and

$$\text{(I.14)}\quad B = (2i)(-1)^m \left[\frac{\Gamma(n-m+1)\,\Gamma(\tfrac{1}{2}-n)\,\Gamma(n+2p+\tfrac{5}{2})}{4\left(\Gamma(n+2p+1-t+1)\,\Gamma(n+\tfrac{3}{2})\,\Gamma(-n-2p-\tfrac{1}{2})\right)} \left(\frac{\gamma}{4}\right)^{-2p-2} \cdot Qs^{-t}_{n+2p+1}(\xi;\gamma^2)\, Ps^{m'}_{n}(\xi;\gamma^2) \right.$$

$$\left. + \frac{\Gamma(n+2p+1+t+1)\,\Gamma(n+\tfrac{3}{2})\,\Gamma(-n-2p-\tfrac{1}{2})}{4\left(\Gamma(n+t+1)\,\Gamma(\tfrac{1}{2}-n)\,\Gamma(n+2p+\tfrac{5}{2})\right)} \left(\frac{\gamma}{4}\right)^{2p} \cdot Ps^{-t}_{n+2p+1}(\xi;\gamma^2)\, Qs^{m'}_{n}(\xi;\gamma^2) \right],$$

where B stands for

$$\left[S^{t(3)}_{n+2p+1}(\xi)\, S^{m(4)'}(\xi) - S^{t(4)}_{n+2p+1}(\xi)\, S^{m(3)'}_{n}(\xi) \right] .$$

Therefore (I.1) and (I.2) become

$$\text{(I.15)}\qquad a^t_{n,p}\ \gamma e^{i(p+1)\pi}\ A = a^t_{n,p}(\xi)\ [4,3] ,$$

$$\text{(I.16)}\qquad a^t_{n,p}\ \gamma e^{i(p+1)\pi}\ B = b^t_{n,p}(\xi)\ [4,3] ,$$

with A and B given by the expressions (I.13) and (I.14) respectively.

Now we are in a position to take the limit $\gamma = 0$. The spheroidal function $Qs_{\nu}^{-t}(\xi;\gamma^2)$ and $Ps_n^m(\xi;\gamma^2)$ go over to the associated Legendre functions $Q_{\nu}^{-t}(\xi)$ and $P_n^m(\xi)$. Hence (3.14) and (3.15) reduce to

$$
\text{(I.17)}\quad \lim_{\gamma \to 0} \Bigg[a_{n,p}^{-t}\, \gamma^2 e^{+i(p+1)\pi} (2i)(-1)^m \Bigg\{ \frac{\Gamma(n-m+1)\,\Gamma(\frac{1}{2}-n)\,\Gamma(n+2p+\frac{5}{2})}{4\left(\Gamma(n+2p+1-m+1)\,\Gamma(n+\frac{3}{2})\,\Gamma(-n-2p-\frac{1}{2})\right)} \left(\frac{\gamma}{4}\right)^{-2p-2} \cdot Q_{n+2p+1}^{-m}(\xi)\, P_n^m(\xi)
$$

$$
+ \frac{\Gamma(n+2p+1+m+1)\,\Gamma(n+\frac{3}{2})\,\Gamma(-n-2p-\frac{1}{2})}{4\left(\Gamma(n+m+1)\,\Gamma(\frac{1}{2}-n)\,\Gamma(n+2p+\frac{5}{2})\right)} \left(\frac{\gamma}{4}\right)^{2p} \cdot P_{n+2p+1}^{-m}(\xi)\, Q_n^m(\xi) \Bigg\} \Bigg]
$$

$$
= -\frac{2i}{e} \int_{-1}^{1} \frac{\eta}{\xi^2 - \eta^2}\, P_n^{m\,\prime}(\eta)\, P_{n+2p+1}^{-m}(\eta)\, d\eta,
$$

and

$$(I.18)\quad \lim_{\gamma\to 0}\Bigg[\alpha^{m}_{n,p}\,\gamma^{2}e^{+i(p+1)\pi}\,2i(-1)^{m}$$

$$\Bigg\{\frac{\Gamma(n-m+1)\,\Gamma(\tfrac{1}{2}-n)^{n}\,\Gamma(n+2p+\tfrac{5}{2})}{4\Big(\Gamma(n+2p+1-m+1)\,\Gamma(n+\tfrac{3}{2})\,\Gamma(-n-2p-\tfrac{1}{2})\Big)}\left(\frac{\gamma}{4}\right)^{-2p-2}$$

$$\cdot\, Q^{-m}_{n+2p+1}(\xi)\,P^{+m'}_{n}(\eta)$$

$$+\frac{\Gamma(n+2p+1+m+1)\,\Gamma(n+\tfrac{3}{2})\,\Gamma(-n-2p-\tfrac{1}{2})}{4\Big(\Gamma(n+m+1)\,\Gamma(\tfrac{1}{2}-n)\,\Gamma(n+2p+\tfrac{5}{2})\Big)}\left(\frac{\gamma}{4}\right)^{2p}$$

$$\cdot\, P^{-m}_{n+2p+1}(\xi)\,Q^{m'}_{n}(\xi)\Bigg\}\Bigg]$$

$$=\frac{2i\xi}{\xi^{2}-1}\int_{-1}^{1}\frac{(1-\eta^{2})}{\xi^{2}-\eta^{2}}\,P^{m'}_{n}(\eta)\,P^{-m}_{n+2p+1}\,d\eta\,.$$

It should be remarked that $\alpha^{m}_{n,p}$ is a function of γ^{2}, as can be seen from Eq. (3.7) or (3.11). It is evident that the coefficients $a^{m}_{n,2r}(\gamma^{2})$ and $a^{-m}_{n+2p+1,2r-2p}(\gamma^{2})$ are proportional to γ^{2r} and γ^{2r-2p}; hence the first terms of the left-hand sides of (I.17) and (I.18) are independent of γ. The same argument applies to the second terms of these expressions. It suffices to show that (I.17) and (I.18) agree with the results derived in Section 4 for $p = 0$ and $p = -1$. In both these cases $\alpha^{m}_{n,p}$ is independent of γ, that is, $\alpha^{m}_{n,p}(\gamma^{2}) = \alpha$, and is given by the integral

$$\text{(I.19)}\quad \alpha^m_{n,-1} = \alpha_{0,-1} = \frac{1}{c}\int_{-1}^{1} \eta\; P^m_n(\eta)\, P^{-m}_{n\pm 1}(\eta)\, d\eta .$$

We get

$$\text{(I.20)}\quad \alpha^m_{n,0}\; e^{i\pi}(2i)(-1)^m \frac{\Gamma(n-m+1)\,\Gamma(\frac{1}{2}-n)\,\Gamma(n+\frac{5}{2})}{\Gamma(n+2p+1-m+1)\,\Gamma(n+\frac{3}{2})\,\Gamma(-n-\frac{1}{2})}\; 4\; Q^{-m}_{n+1}(\xi)\, P^m_n(\xi)$$

$$= \frac{2i}{c}\int_{-1}^{1} \frac{\eta}{\xi^2-\eta^2}\, P^m_n(\eta)\, P^{-m}_{n+1}\, d\eta ,$$

$$\text{(I.21)}\quad \alpha^m_{n,-1}\; e^{i\pi}(2i)(-1)^m \frac{\Gamma(n+m)\,\Gamma(n+\frac{3}{2})\,\Gamma(-n-\frac{5}{2})}{\Gamma(n+m+1)\,\Gamma(\frac{1}{2}-n)\,\Gamma(n+\frac{1}{2})}\; 4\; P^{-m}_{n+1}(\xi)\, Q^m_n(\xi)$$

$$= \frac{2i}{c}\int \frac{\eta}{\xi^2-\eta^2}\, P^m_n(\eta)\, P^{-m}_{n-1}(\eta)\, d\eta .$$

To prove that (I.20) and (I.21) are the correct integral relations between products of associated Legendre functions one must show that for large ξ they reduce to identities in ξ. For this purpose we have only to consider the leading terms of $Q^m_n(\xi)$, $P^m_n(\xi)$, etc., and compare the coefficients of both sides of (I.20) and (I.21) in ξ. Then the statement is easily verified*.

In the second limiting case, $\gamma = 0\ \frac{z}{\gamma} = \xi$, the results are rather trivial, since the constants $\alpha^m_{n,p}$ vanish for all $p \neq 0$ and -1 (see Sect. 3, Eq. (3.7)). When we put $p = 0$ and $p = -1$ and remember the limiting expressions for $S^{\mu(j)}_\nu(\xi,\gamma^2)$ and $ps^\mu_\nu(\eta;\gamma^2)$, we find from (3.12)

---

* We remark that for $p = 0$, only the first terms of (I.17) and (I.18) give a non-vanishing result, and for $p = -1$, only the second terms remain, as can be easily seen.

$$-\Psi_{n+1}^{(j)}(z) = \Psi_n^{(j)'}(z) - \frac{n}{z}\Psi_n^{(j)}(z) , \tag{I.21}$$

$$-\Psi_{n-1}^{(j)}(z) = \Psi_n^{(j)}(z) + \frac{(n+1)}{z}\Psi_n^{(j)}(z) , \tag{I.23}$$

where $\Psi_n^{(j)}(z)$ is defined in (4.19). By adding and subtracting the above relations, we find, respectively,

$$\Psi_{n-1}^{(j)}(z) + \Psi_{n+1}^{(j)}(z) = \frac{2n+1}{z}\Psi_n^{(j)}(z),$$

$$\Psi_{n-1}^{(j)}(z) - \Psi_{n+1}^{(j)}(z) = 2\Psi_n^{(j)'}(z) + \frac{1}{z}\Psi_n^{(j)}(z),$$

which are the well-known recurrence relations between three contiguous cylindrical functions [10] .

Appendix II. Useful formulas involving associated Legendre functions

We list here the following formulas, which have been employed in the text:

$$\text{(II.1)} \qquad \sqrt{1-\eta^2}\; P_n^m(\eta) = \frac{1}{2n+1}\left(P_{n+1}^{m+1}(\eta) - P_{n-1}^{m+1}(\eta)\right)$$

$$\text{(II.2)} \qquad \sqrt{1-\eta^2}\; P_n^m(\eta) = \frac{1}{2n+1}\left((n-m+1)(n-m+2)\, P_{n+1}^{m-1}(\eta) - (n+m)(n+m-1) P_{n-1}^{m-1}(\eta)\right)$$

$$\text{(II.3)} \qquad \sqrt{1-\eta^2}\; P_n^{m\,\prime}(\eta) = \frac{1}{2}\left[(n+m)(n-m+1)\, P_n^{m-1}(\eta) - P_n^{m+1}(\eta)\right]$$

$$\text{(II.4)} \qquad (1-\eta^2)\, P_n^{m\,\prime}(\eta) = \frac{1}{2n+1}\left[(n+1)(n+m)\, P_{n-1}^{m}(\eta) - n(n-m+1)\, P_{n+1}^{m}(\eta)\right]$$

$$\text{(II.5)} \qquad \sqrt{1-\eta^2}\, P_n^{m\,\prime}(\eta) + \frac{m\eta}{\sqrt{1-\eta^2}}\, P_n^m(\eta) = -\, P_n^{m+1}$$

$$\text{(II.6)} \qquad \sqrt{1-\eta^2}\, P_n^{m\,\prime}(\eta) + \frac{m\eta}{\sqrt{1-\eta^2}}\, P_n^m(\eta) = (n+m)\,(n-m+1)\, P_n^{m-1}\,.$$

From (II.1) one can write immediately the expression for $(1-\eta^2)\, P_n^m(\eta)$:

$$\text{(II.7)} \qquad (1-\eta^2)\, P_n^m(\eta) = \frac{1}{(2n+1)(2n+3)}\, P_{n+2}^{m+2}(\eta) - \frac{2}{(2n+3)2n-1}\, P_n^{m+2}(\eta)$$

$$+ \frac{1}{2n+1}\,\frac{1}{2n-1}\, P_{n-1}^{m+2}(\eta)$$

and by induction one obtains the general case

$$(\text{II.8}) \quad (1-\eta^2)^p P_n^m(\eta) = \sum_{q=0}^{q=p} (-1)^q \frac{(2n-2q-1)!!(2n+2q-4q+1)\binom{p}{q}}{(2n+2p-2q+1)!!} P_{n+p-2q}^{m+p}(\eta)$$

where

$$(2n)!! = (2n+1)\ (2n-1)\ (2n-3)\ \ldots ,$$

and where from (II.2)

$$(\text{II.9}) \quad (1-\eta^2)^p P_n^m(\eta) = \sum_{q=0}^{q=p} (-1)^q \frac{(n+m)!(n+2p-2q-m)!}{(n-m)!(n+m-2q)!}$$

$$\cdot \frac{(2n-2q-1)!!}{(2n+2p-2q+1)!!} (2n+2q-4q+1)\ P_{n+p-2q}^{m-p}(\eta) \ .$$

From (II.8) and (II.9) one can easily evaluate the integrals

$$(\text{II.10}) \quad \int_{-1}^{1} \eta^q \sqrt{1-\eta^2}\ P_n^m(\eta)\ P_{n+2p+1}^{-(m+1)}(\eta)\ d\eta$$

$$= 0 \qquad \text{for } q \neq 2p$$

$$= \frac{2^{2p+2}\ \Gamma(2n+1)(n+2p+1)!(n+2p-m)!}{n!\ (n-m)!\ (2n+4p+3)!} \qquad \text{for } q = 2p$$

$$(\text{II.11}) \quad \int_{-1}^{1} \eta^q \sqrt{1-\eta^2}\ P_n'^m(\eta)\ P_{n+2p+1}^{-(m+1)}(\eta)\ d\eta$$

$$= 0 \qquad \text{for } q \neq 2p+1$$

$$= \frac{-2^{2p+2}\ \Gamma(2n)(n+2p+1)!(n+2p-m)!}{(n-1)!\ (n-m)!(2n+4p+3)!} \qquad \text{for } q = 2p+1$$

With the aid of (II.8) and (II.9) one can evaluate integrals involving products of three associated Legendre functions.

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